PG – 623

IV Semester M.Sc. Examination, June 2017
(RNS – Repeaters) (2011-12 and Onwards)
MATHEMATICS
M 401 : Measure and Integration

Time : 3 Hours Max. Marks : 80

Instructions : *i) Answer **any five** questions choosing atleast **two** from **each** Part.*
*ii) **All** questions carry **equal** marks.*

PART – A

1. a) Prove that in any algebra a over a set X, a countable union can be expressed as a countable disjoint union.
   b) Define σ-algebra. Prove that every algebra on a set X is contained in the smallest σ-algebra on X.
   c) Define an outer measure. Prove that outer measure is translation invariant.
   (6+6+4)

2. a) Show that the Lebesgue measure of an interval is equal to its length. Hence prove that, if A is a countable set then $m^*A = 0$.
   b) Define G_σ – sets. Let E be any set. Then prove that
      i) Given $\varepsilon > 0$, there exists an open set $O \supset E$ such that $m^*(0) < m^*E + \varepsilon$.
      ii) There exist a G_σ-set $G \supset E$ such that $m^*(E) = m^*(G)$. **(8+8)**

3. a) Let $\{E_n\}$ be a decreasing sequence of measurable sets with $mE_i < \infty$. Then prove that

   $$m\left(\bigcap_{i=1}^{\infty} E_i\right) = \underset{n\to\infty}{\text{lt}}\, m\,E_n$$

   Give an example to show that the condition $mE_i < \infty$ can not be released in the above case.
   b) Let $\{E_i\}$ be an infinite increasing sequence of measurable sets. Then prove that $m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} mE_n$. **(8+8)**

P.T.O.

4. a) Define a measurable function. Show that the following statements are equivalent for a function $f: E \to R^*$ where R^* denotes the extended real number system.

   i) $\{x \in E / f(x) > a\}$ is measurable, $\forall a \in R$

   ii) $\{x \in E / f(x) \geq a\}$ is measurable, $\forall a \in R$

   iii) $\{x \in E / f(x) < a\}$ is measurable, $\forall a \in R$

   iv) $\{x \in E / f(x) \leq a\}$ is measurable, $\forall a \in R$

   Further show that the above statements imply that for any $b \in R^*$, $\{x \in E / f(x) = b\}$ is measurable.

   b) Let f be an extended real valued function defined on a measurable set E. Then prove that f is measurable if and only if for every open set G in $\mathbb{R}$, the set f^{-1} (G)is measurable. **(8+8)**

PART – B

5. a) Define a simple function. Let ϕ and ψ be simple functions which vanish outside a set of finite measure. Then prove the following :

   i) $\int a\phi + b\psi = a\int \phi + b\int \psi \quad a, b \in \mathbb{R}$

   ii) If $\phi \geq \psi$ a.e then $\int \phi \geq \int \psi$.

   b) Prove that a bounded function defined on a measurable set E of finite measure is Lebesgue integrable if and only if it is measurable. **(8+8)**

6. a) Let f and g be two non-negative measurable functions. If f is integrable over E and $g(x) \leq f(x)$ on E. Then prove that g is integrable on E and $\int_E f - g \leq \int_E f - \int_E g$.

   b) State and prove bounded convergence theorem.

   c) Define Dini derivatives of a function. Let f(x) be a function defined by f(0) = 0 and $f(x) = x \sin\left(\frac{1}{x}\right)$ for $x \neq 0$. Find $D^+f(0)$, $D_+f(0)$, $D^-f(0)$, $D_-f(0)$. **(5+6+5)**

7. a) If f is integrable on [a, b] and $\int_a^x f(t).dt = 0$ for all $x \in [a, b]$ then prove that $f(t) = 0$ a.e on [a, b].

   b) Define absolute continuous function. If f(x) and g(x) are absolutely continuous functions then prove that $f \pm g$, $f.g$, $\frac{f}{g}(g \neq 0)$ are also absolutely continuous.

   c) Show that a monotonic function on [a, b] is of bounded variation. **(6+6+4)**

8. a) Establish Minkowski's inequality.

   b) Prove that a normal linear space is complete if and only if every absolutely summable series is summable. **(8+8)**

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